

# Non-integral central extensions of loop groups

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**ABSTRACT.** It is well-known that the central extensions of the loop group of a compact, simple and 1-connected Lie group are parametrised by their level  $k \in \mathbb{Z}$ . This article concerns the question how much can be said for arbitrary  $k \in \mathbb{R}$  and we show that for each  $k$  there exists a Lie groupoid which has the level  $k$  central extension as its quotient if  $k \in \mathbb{Z}$ . By considering categorified principal bundles we show, moreover, that the corresponding Lie groupoid has the expected bundle structure.

## Introduction

In this paper we generalise a construction of the universal central extension  $\widehat{\Omega K}$  of the loop group  $\Omega K$  of a compact simple and 1-connected Lie group  $K$ , going back to Mickelsson and Murray [Mic87], [Mur88]. They construct  $\widehat{\Omega K}$  as a quotient of a central extension  $U(1) \times_{\kappa} P_e \Omega K$  of the based path group  $P_e \Omega K$ . For this construction one has the freedom to choose a real number  $k \in \mathbb{R}$  (after having fixed all normalisations appropriately), which is usually referred to as the *level*. The construction from [Mic87] and [Mur88] then yields a normal subgroup  $N_k \trianglelefteq U(1) \times_{\kappa} P_e \Omega K$  if and only if  $k \in \mathbb{Z}$  and constructs  $\widehat{\Omega K}$  as  $(U(1) \times_{\kappa} P_e \Omega K)/N_1$ .

The point of this article is that, although the construction of  $N_k$  works if and only if  $k \in \mathbb{Z}$ , for general  $k$  there still exists an infinite-dimensional Lie group  $\mathcal{K}$  acting on  $U(1) \times_{\kappa} P_e \Omega K$  and the quotient of this action coincides with  $(U(1) \times_{\kappa} P_e \Omega K)/N_k$  if  $k \in \mathbb{Z}$ . This then gives rise to an action Lie groupoid. By passing to a Morita equivalent Lie groupoid we show that this Lie groupoid has the structure of a generalised principal bundle.

The results that we get here are closely related to the general extension theory of infinite-dimensional Lie groups by categorical Lie groups from [Woc08]. However, this article concerns more the global and differential point of view to those extensions for the particular case of loop groups, while [Woc08] provides a more detailed perspective from the side of cocycles.

**Notation:** Throughout this article,  $G$  denotes a (possibly infinite-dimensional) connected Lie group with Lie algebra  $\mathfrak{g}$ , modelled on a locally convex space,  $\mathfrak{z}$  is a sequentially complete locally convex space and  $\Gamma \subseteq \mathfrak{z}$  is a discrete subgroup of (the additive group of)  $\mathfrak{z}$ . Moreover, we set  $Z := \mathfrak{z}/\Gamma$ .

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Most of the time,  $G$  will be the pointed loop group

$$\Omega K := \{\gamma : \mathbb{R} \rightarrow K : \gamma(0) = e, \gamma(x+n) = \gamma(x) \forall n \in \mathbb{N}\}$$

of smooth and pointed loops in a compact, simple and 1-connected Lie group  $K$ , endowed with the usual Fréchet topology and point-wise multiplication. The Lie algebra of  $\Omega K$  is then

$$\Omega \mathfrak{k} := \{\gamma : \mathbb{R} \rightarrow \mathfrak{k} : \gamma(0) = 0, \gamma(x+n) = \gamma(x) \forall n \in \mathbb{N}\}$$

with  $\mathfrak{k} := L(K)$ . In this case,  $\mathfrak{z}$  will be  $\mathbb{R}$ ,  $\Gamma$  will be  $\mathbb{Z}$  and thus  $Z = \mathbb{R}/\mathbb{Z} =: U(1)$ . We circumvent all normalisation issues by choosing this quite unnatural realisation of the circle group. Moreover, we denote by  $\exp$  the canonical quotient map  $\exp : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ .

We will be a bit sloppy in our conventions concerning the precise model for  $S^1$  and  $B^2$ . Instead, we collect the things that we want to assume:

- $B^2$  and  $S^1$  are manifolds with corners such that  $S^1$  may be identified with a submanifold of  $B^2$  (which we denote by  $\partial B^2$ ) and the base-point of  $B^2$  is contained in  $\partial B^2$ .
- $C_*^\infty(S^1, G)$  may be identified with the kernel of the evaluation map  $\text{ev} : P_e G \rightarrow G$ , where  $P_e G$  denotes the space of smooth maps  $f : [0, 1] \rightarrow G$  with  $f(0) = e$ .
- The map  $C_*^\infty(B^2, G) \rightarrow C_*^\infty(S^1, G)_e$ ,  $f \mapsto f|_{\partial B^2}$  is surjective.

Here, the subscript  $*$  denotes pointed maps and the subscript  $e$  denotes the connected component of the identity.

## 1. Generalities on central extensions of infinite-dimensional Lie groups

We briefly review essentials on central extensions of infinite-dimensional Lie group, established by Neeb in [Nee02]. There, the second locally smooth group cohomology  $H_{\text{loc}}^2(G, Z)$  is defined to be the set of functions  $f : G \times G \rightarrow Z$  such that

- $f$  is smooth on  $U \times U$  for  $U \subseteq G$  some open identity neighbourhood
- $f(g, h) + f(gh, k) = f(g, hk) + f(h, k)$  for all  $g, h, k \in G$
- $f(g, e) = f(e, g) = 0$  for all  $g \in G$ ,

(called *locally smooth group cocycles* in this paper) modulo the equivalence relation

$$(1.1) \quad (f \sim f') : \Leftrightarrow f(g, h) - f'(g, h) = b(g) - b(gh) + b(h)$$

for some  $g : G \rightarrow Z$  which is smooth on some identity neighbourhood and satisfies  $b(e) = 0$ . Similarly, we define  $H_{\text{glob}}^2(G, Z)$  to be defined in the same way except that we require  $f$  (respectively  $b$ ) to be smooth on  $G \times G$  (respectively  $G$ ). We shall call such a  $f$  a *globally smooth group cocycle*. Then in [Nee02] it is shown that  $H_{\text{loc}}^2(G, Z)$  corresponds bijectively to the equivalence classes of central extensions

$$(1.2) \quad Z \rightarrow \widehat{G} \rightarrow G$$

of Lie groups such that (1.2) is a *locally* trivial principal bundle and that  $H_{\text{glob}}^2(G, Z)$  corresponds bijectively to equivalence classes of central extensions of Lie groups such that (1.2) is a *globally* trivial principal bundle.

The bulk of the work in [Nee02] concerns the integration issue for central extensions, i.e., how to derive a continuous Lie algebra cocycle  $D(f) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{z}$  from a locally smooth group cocycle and to determine whether for a given continuous Lie

algebra cocycle  $\omega$  there exists a locally smooth group cocycle  $f$  such that  $[Df] = [\omega] \in H_c^2(\mathfrak{g}, \mathfrak{z})$  (where the subscript  $c$  means continuous Lie algebra cohomology). In the latter case we say that  $\omega$  *integrates*. The main result in [Nee02] is an exact sequence

$$\begin{aligned} \mathrm{Hom}(\pi_1(G), Z) \rightarrow H^2(G, Z) \xrightarrow{D} H_c^2(\mathfrak{g}, \mathfrak{z}) \\ \xrightarrow{P} \mathrm{Hom}(\pi_2(G), Z) \oplus \mathrm{Hom}(\pi_1(G), \mathrm{Lin}_c(\mathfrak{g}, \mathfrak{z})), \end{aligned}$$

where  $\mathrm{Lin}_c$  denotes continuous linear maps and

$$\mathrm{per}_\omega := \mathrm{pr}_1(P([\omega])) : \pi_2(G) \rightarrow Z, \quad [\sigma] \mapsto \left[ \int_\sigma \omega^l \right]$$

for  $\sigma$  a smooth representative of  $[\sigma] \in \pi_2(G)$  and  $\omega^l$  the left-invariant 2-form on  $G$  with  $\omega^l(e) = \omega$ . In particular, when  $G$  is simply connected, then the sequence reduces to a shorter exact sequence

$$0 \rightarrow H^2(G, Z) \xrightarrow{D} H_c^2(\mathfrak{g}, \mathfrak{z}) \xrightarrow{\mathrm{per}} \mathrm{Hom}(\pi_2(G), Z).$$

Thus a given cocycle  $\omega$  integrates in this case if and only if the corresponding *period homomorphism*  $\mathrm{per}_\omega$  vanishes.

## 2. The topological type of central extensions

In [Mic85], Mickelsson derives a Čech 1-cocycle for  $\widehat{\Omega \mathrm{SU}_2}$ . In this section we shall describe how to derive the topological type of the principal bundle

$$Z \rightarrow \widehat{G} \rightarrow G$$

for a central extension coming from a locally smooth cocycle  $f : G \times G \rightarrow Z$ . This description is much more general than the one from [Mic85] and it will become apparent from this construction that for a globally smooth cocycle the corresponding bundle is automatically trivial. For this we will make use of the following fact.

**THEOREM 2.1.** *Let  $H$  be a group  $W \subseteq H$  be a subset containing  $e$  and let  $W$  be endowed with a manifold structure. Moreover, assume that there exists an open neighbourhood  $Q \subseteq W$  of  $e$  with  $Q^{-1} = Q$  and  $Q \cdot Q \subseteq W$  such that*

- $Q \times Q \ni (g, h) \mapsto gh \in W$  is smooth,
- $Q \ni g \mapsto g^{-1} \in Q$  is smooth and
- $Q$  generates  $H$  as a group.

*Then there exists a manifold structure on  $H$  such that  $Q$  is open in  $H$  and such that group multiplication and inversion is smooth. Moreover, for each other choice of  $Q$ , satisfying the above conditions, the resulting smooth structures on  $H$  coincide.*

**PROOF.** The proof is well-known and straight-forward, cf. [Woc08, Thm. II.1], [Bou98, Prop. III.1.9.18].  $\square$

We now derive a central extension from a locally smooth group cocycle  $f : G \times G \rightarrow Z$ . First, we define a twisted group structure on the set-theoretical direct product  $Z \times G$  by  $(a, g) \cdot (b, h) := (a + b + f(g, h), gh)$ . Then the requirement on  $f$  to define a group cocycle implies that this defines a group multiplication with neutral element  $(0, e)$  and  $(a, g)^{-1} = (-a - f(g, g^{-1}), g^{-1})$ . We denote this group by  $Z \times_f G$ . If  $f$  is smooth on  $U \times U$  and  $V \subseteq U$  is an open identity neighbourhood with  $V \cdot V \subseteq W$  and  $V^{-1} = V$ , then  $Z \times U$  carries the product manifold structure

and  $Z \times V$  is open in  $Z \times U$ . Since  $G$  is assumed to be connected,  $Z \times_f G$  is generated by  $Z \times V$  and the preceding theorem yields a Lie group structure on  $Z \times_f G$ . Clearly, the sequence

$$Z \rightarrow Z \times_f G \rightarrow G$$

is a locally trivial principal bundle for we have the smooth section

$$U \ni x \mapsto (0, x) \in Z \times U \subseteq Z \times_f G.$$

LEMMA 2.2. *The assignment*

$$\tau(f)_{g,h} : gV \cap hV \rightarrow Z, \quad x \mapsto f(g, g^{-1}x) - f(h, h^{-1}x)$$

defines a Čech 1-cocycle  $\tau(f)$  on the open cover  $(gV)_{g \in G}$  and thus an element of  $\check{H}^1(G, Z)$ . If  $[f] = [f']$  in  $H_{\text{loc}}^2(G, Z)$ , then  $[\tau(f)] = [\tau(f')]$  in  $\check{H}^1(G, Z)$ .

PROOF. We first note that  $gh^{-1} \in V \cdot V \subseteq W$  if  $gV \cap hV \neq \emptyset$ . From this it follows that

$$x \mapsto f(g, g^{-1}x) - f(h, h^{-1}x) = f(g^{-1}h, h^{-1}x) - f(g, g^{-1}h)$$

is smooth on  $gV \cap hV$ , for  $f$  is smooth on  $U \times U$ . From the definition it is also clear that  $\tau(f)_{g,h} - \tau(f)_{g,k} + \tau(f)_{h,k}$  vanishes.

If  $[f] = [f']$ , then  $f(g, h) - f'(g, h) = b(g) - b(gh) + b(h)$  for  $b : G \rightarrow Z$ . We assume without loss of generality that  $f$  and  $f'$  are smooth on  $U \times U$  and  $b$  is smooth on  $U$  (presumably, the identity neighbourhoods may be distinct for  $f$  and  $f'$  and  $b$ ). Then

$$\tau(b)_g : gV \rightarrow Z, \quad x \mapsto b(g) + b(g^{-1}x)$$

defines a Čech cochain with  $\tau(f) - \tau(f') = \check{\delta}(\tau(b))$ .  $\square$

We thus have a map  $\tau : H_{\text{loc}}^2(G, Z) \rightarrow \check{H}^1(G, Z)$ , which clearly is a group homomorphism.

PROPOSITION 2.3. *The principal bundle*

$$(2.1) \quad Z \rightarrow Z \times_f G \rightarrow G$$

is classified by  $[\tau(f)] \in \check{H}^1(G, Z)$ .

PROOF. From the construction of the topology on  $Z \times_f G$  it follows immediately that  $\sigma_e(x) := (0, x)$  defines a smooth section on  $V$ . Thus the assignment

$$\sigma_g : gV \rightarrow Z \times_f G, \quad x \mapsto (f(g, g^{-1}x), x) = (\lambda_{(0,g)} \circ \sigma_e \circ \lambda_{g^{-1}})(x)$$

is smooth, where  $\lambda_{g^{-1}}$  denotes left multiplication in  $G$  with  $g^{-1}$  and  $\lambda_{(0,g)}$  denotes left multiplication in  $Z \times_f G$  with  $(0, g)$ . Consequently,  $(\sigma_g : gV \rightarrow Z \times_f G)_{g \in G}$  defines a system of sections for the principal bundle (2.1) and since  $\tau(f)$  satisfies  $\sigma_g(x) = \sigma_h(x) \cdot \tau(f)_{g,h}(x)$ , this already shows the claim.  $\square$

COROLLARY 2.4. *A locally smooth cocycle  $f : G \times G \rightarrow Z$  is equivalent to a globally smooth cocycle if and only if the principal bundle, underlying  $Z \rightarrow Z \times_f G \rightarrow G$ , is topologically trivial.*

PROOF. If the bundle is topologically trivial, then there exists a smooth section  $\sigma : G \rightarrow Z \times_f G$  and

$$f'(g, h) := \sigma(g)\sigma(h)\sigma(gh)^{-1}$$

defines a  $Z$ -valued cocycle. Since  $Z$  acts freely on  $Z \times_f G$ , we have  $(0, g) = \sigma(g) \cdot b(g)$  for  $b : G \rightarrow Z$ , smooth on a identity neighbourhood and satisfying (1.1). The “only if” part is clear from the construction of  $\tau(f)$ .  $\square$

We thus obtain a sequence

$$0 \rightarrow H_{\text{glob}}^2(G, Z) \rightarrow H_{\text{loc}}^2(G, Z) \xrightarrow{\tau} \check{H}^1(G, Z)$$

which is obviously exact. It would be interesting to determine for which groups and which coefficients the map  $\tau$  is *not* surjective. Note that the case of  $Z$  being connected is the interesting one, since for a discrete group  $A$ , each principal  $A$ -bundle over  $G$  is a covering and thus admits a compatible Lie group structure.

### 3. The universal central extension of Loop groups

The results described in the preceding section applies to loop groups  $\Omega K$  in the following way. If  $\langle \cdot, \cdot \rangle : \mathfrak{k} \times \mathfrak{k} \rightarrow \mathbb{R}$  denotes the Killing form (which is non-degenerate and negative definite in our case), then

$$\omega : \Omega \mathfrak{k} \times \Omega \mathfrak{k} \rightarrow \mathbb{R}, \quad (f, g) \mapsto \int_{S^1} \langle f(t), g'(t) \rangle dt$$

defines a continuous Lie algebra cocycle. If we normalise  $\langle \cdot, \cdot \rangle$  in such way that the left-invariant extension  $\omega^l$  of  $\omega$  satisfies  $\int_{\sigma} \omega^l = 1$  for  $\sigma$  a generator<sup>1</sup> of  $\pi_2(\Omega K) \cong \pi_3(K) \cong \mathbb{Z}$ , then the calculations in [MN03] show that for  $Z = \mathbb{R}$  we have  $\text{per}_{\omega}(\pi_2(\Omega K)) = \mathbb{Z}$ . Thus the cocycle  $k \cdot \omega$  integrates to a locally smooth group cocycle  $f_k : \Omega K \times \Omega K \rightarrow U(1)$ , defining a central extension

$$U(1) \rightarrow \widehat{\Omega K}_k \rightarrow \Omega K$$

if and only if  $k \in \mathbb{Z}$ . Moreover, the central extension for  $k = \pm 1$  is universal, as it is shown in [MN03]. This means that for each other central extension  $Z \rightarrow \widehat{\Omega K} \rightarrow \Omega K$  there exist unique morphisms  $U(1) \rightarrow Z$  and  $\widehat{\Omega K}_{\pm 1} \rightarrow \widehat{\Omega K}$  making the diagram

$$\begin{array}{ccccc} U(1) & \longrightarrow & \widehat{\Omega K}_{\pm 1} & \longrightarrow & \Omega K \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & \widehat{\Omega K} & \longrightarrow & \Omega K \end{array}$$

commute.

There also exist more ad-hoc constructions of  $\widehat{\Omega K}_k$ , cf. [PS86], [Mic87], [Mur88], [MS01] or [MS03] which, more or less, all construct  $\widehat{\Omega K}_k$  first constructing a central extension  $\widehat{P_e \Omega K} \rightarrow P_e \Omega K$ , corresponding to the pull-back of  $\widehat{\Omega K}_k$  along  $\text{ev} : P_e \Omega K \rightarrow \Omega K$ , and then considering an appropriate quotient of  $\widehat{P_e \Omega K}$ . Since  $P_e \Omega K$  is contractible, the results of the preceding sections imply that the pull-back of the central extension of  $L(\Omega K) \cong \Omega \mathfrak{k}$  along the evaluation homomorphism  $L(\text{ev}) : L(P_e \Omega K) \rightarrow L\mathfrak{k}$  to  $L(P_e \Omega K) \cong P_e \Omega \mathfrak{g}$  integrates to a central extension

$$U(1) \rightarrow \widehat{P_e \Omega K} \rightarrow P_e \Omega K,$$

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<sup>1</sup>The ambiguity in the sign that one still has for the normalisation of  $\langle \cdot, \cdot \rangle$  will play no role in the sequel.

given by a globally smooth cocycle  $\kappa : PG \times PG \rightarrow U(1)$ . The constructions of  $\widehat{\Omega K}$  cited above all deal with an explicit description of a normal subgroup  $N_k \trianglelefteq \widehat{P_e \Omega K}$  in order to obtain an induced central extension

$$U(1) \rightarrow \underbrace{\widehat{P_e \Omega K} / N_k}_{\cong \widehat{\Omega K}_k} \rightarrow \underbrace{P_e \Omega K / \Omega(\Omega K)}_{\cong \Omega K}$$

(cf. also [GN03, Section III]).

#### 4. Central extensions of loop groups from Lie groupoids

We shall put more structure on the ad-hoc construction of  $\widehat{\Omega K}$  from the previous section. In particular, we show that  $\widehat{P_e \Omega K} / N_k$  may be obtained as the quotient of an action Lie groupoid, which also exists for non-integral values of  $k$ .

REMARK 4.1. We briefly recall the essential notions for Lie groupoids. A Lie groupoid is a category object in the category of locally convex manifolds, such that source and target maps admit local inverses<sup>2</sup>. More precisely, it consists of two locally convex manifolds  $M_1$  and  $M_0$ , together with smooth maps  $\text{id} : M_0 \rightarrow M_1$  and  $s, t : M_1 \rightarrow M_0$ , admitting local inverses, and a smooth map<sup>3</sup>  $\circ : M_1 \times_t M_1 \rightarrow M_0$  satisfying the usual relations of identity, source, target and composition map of a small category. Moreover, we require that each morphism of this category is invertible and that the map  $M_1 \rightarrow M_1$ , assigning to each morphism its inverse, is smooth. The quotient of such a Lie groupoid is defined to be the set of equivalence classes of isomorphic objects. The smooth structure on  $M_0$  may or may not induce a smooth structure on the quotient, depending on how badly the quotient actually is behaved.

A typical example of a Lie groupoid, called *action groupoid*, is obtained from a smooth right action of a Lie group  $G$  on a manifold  $M$ . With this data given, we set  $M_0 := M$ ,  $M_1 := M \times G$ ,  $\text{id}(m) := (m, e)$ ,  $s(m, g) := m$ ,  $t(m, g) := m \cdot g$  and  $(m \cdot g, h) \circ (m, g) := (m, g \cdot h)$ . Clearly, the inverse of  $(m, g)$  is  $(m \cdot g, g^{-1})$ . The quotient of the Lie groupoid clearly is given by  $M/G$ . If it admits a smooth structure such that the quotient map  $M \rightarrow M/G$  is smooth and admits local inverses, then the action groupoid is Morita equivalent<sup>4</sup> to the Lie groupoid with  $M_0 = M_1 = M/G$  and all structure maps the identity. If  $M/G$  does not carry a smooth structure, then the action groupoid is an appropriate replacement for  $M/G$ .

In order to motivate our procedure we recall that  $N_k$  is defined to be the subset

$$\{(z, \gamma) \in U(1) \times P_e \Omega K : \gamma \in C_*^\infty(S^1, \Omega K), z = \exp(-k \cdot \int_{D_\gamma} \omega^l)\},$$

where  $D_\gamma : B^2 \rightarrow \Omega K$  is a smooth map with  $D_\gamma|_{\partial B^2} = \gamma$  and  $\omega^l$  is the left-invariant 2-form on  $\Omega K$  with  $\omega^l(e) = \omega$ . Since  $\omega^l$  is an integral 2-form on  $\Omega K$ , the value of

<sup>2</sup>We shall use concepts from the usual theory of Lie groupoids by replacing the term “surjective submersion” at each occurrence by the term “admits local inverses”. This is equivalent in the finite-dimensional case but may not be in the infinite-dimensional one.

<sup>3</sup>Note that the existence of local inverses for  $s$  and  $t$  ensures the existence of a manifold structure on  $M_1 \times_t M_1$ .

<sup>4</sup>Morita equivalent Lie groupoids are the correct replacement for the concept of equivalent categories. In fact, Morita equivalent Lie groupoids are equivalent as categories and possess the same amount of “differential” information, cf. [MM03].

$\exp(k \cdot \int_{D_\gamma} \omega^l)$  does not depend on the choice of  $D_\gamma$  if  $k \in \mathbb{Z}$ . The groupoid that we will construct carries some more information, namely not only the boundary value of  $D_\gamma$  but also the homotopy type of it relative to  $\partial B^2$ . This information is contained in the group

$$C_*^\infty(B^2, \Omega K) / C_*^\infty(S^2, \Omega K)_e,$$

(where we identify  $C_*^\infty(S^2, \Omega K)$  with the normal subgroup in  $C_*^\infty(B^2, \Omega K)$  of functions that vanish on  $\partial B^2$ ) which we shall now endow with a Lie group structure. The following proof shall make use of the fact that smooth and continuous homotopies of functions with values in locally convex manifold agree, we refer to [Woc09a] for details on this.

LEMMA 4.2. *If  $G$  is a connected locally convex Lie group with Lie algebra  $\mathfrak{g}$ , then the quotient group*

$$C_*^\infty(B^2, G) / C_*^\infty(S^2, G)_e$$

*carries a Lie group structure, modelled on  $C_*^\infty(S^1, \mathfrak{g})$ .*

PROOF. We shall make use of the Lie group structure on  $C_*^\infty(M, G)$  (with respect to point-wise group operations), which exists for each compact manifold  $M$ , possibly with corners [Woc06]. If  $U$  is open in  $G$ , then

$$C^\infty(M, U) := \{f \in C_*^\infty(M, G) : f(M) \subseteq U\}$$

is open in  $C^\infty(M, G)$  and, likewise, if  $U'$  is open in  $\mathfrak{g}$ , then  $C^\infty(M, U')$  is open in  $C^\infty(M, \mathfrak{g})$ . If  $U \subseteq G$  is an open identity neighbourhood and  $\varphi : U \rightarrow \varphi(U) \subseteq \mathfrak{g}$  is a chart with  $\varphi(U)$  open and convex and satisfying  $\varphi(e) = 0$ , then a chart for the manifold structure, underlying  $C_*^\infty(M, G)$ , is given by

$$C^\infty(M, U) \ni f \mapsto \varphi \circ f \in C^\infty(M, \varphi(U)).$$

Clearly, this induces a map

$$\tilde{\varphi} : q(C_*^\infty(B^2, U)) \rightarrow C_*^\infty(S^1, \varphi(U)), \quad [f] \mapsto \varphi \circ (f|_{\partial B^2}),$$

where  $q : C_*^\infty(B^2, G) \rightarrow C_*^\infty(B^2, G) / C_*^\infty(S^2, G)_e$  denotes the canonical quotient map. This map is bijective since each map  $f \in C_*^\infty(S^1, \varphi(U))$  has a homotopy to the map which is constantly 0, defining an extension of  $f$  to a map  $F : B^2 \rightarrow \varphi(U)$  with  $F|_{\partial B^2} = f$  and  $[\varphi^{-1} \circ F]$  is mapped to  $f$  under  $\tilde{\varphi}$ . Similarly, we deduce that  $\tilde{\varphi}$  is injective, since each two maps in  $C_*^\infty(B^2, U)$ , which restrict to the same value on  $\partial B^2$ , are homotopic.

We are now ready to verify that the conditions of Theorem 2.1 are satisfied, which we want to apply to the subset  $W := q(C_*^\infty(B^2, U))$ . On this we have a smooth structure, induced by the bijection  $\tilde{\varphi}$ . Moreover, if  $V \subseteq U$  is an open identity neighbourhood of  $G$  with  $V^2 \subseteq U$  and  $V^{-1} = V$ , then  $q(C_*^\infty(B^2, V))$  is open in  $q(C_*^\infty(B^2, U))$ . The structure maps on the mapping group under consideration are all given by the point-wise group structure in  $G$ , and so it follows that the coordinate representation of the structure maps on  $q(C_*^\infty(B^2, V))$  coincides with the coordinate representation of the structure maps of  $C_*^\infty(S^1, G)$ . Since the latter are smooth it follows that the structure maps on  $q(C_*^\infty(B^2, V))$  are smooth. Finally,  $q(C_*^\infty(B^2, V))$  generates  $C_*^\infty(B^2, G) / C_*^\infty(S^2, G)_e$ , because  $G$  is connected.  $\square$

Note that there is a natural homomorphism

$$\mathcal{K} := C_*^\infty(B^2, \Omega K) / C_*^\infty(S^2, \Omega K)_e \rightarrow C_*^\infty(S^1, \Omega K), \quad [f] \mapsto f|_{\partial B^2},$$

which obviously is smooth and surjective, because  $\pi_1(\Omega K)$  vanishes. The kernel of this map is  $C_*^\infty(S^2, \Omega K)/C_*^\infty(S^2, \Omega K)_e \cong \pi_2(\Omega K)$  and we thus obtain a central extension

$$\pi_2(\Omega K) \rightarrow \mathcal{K} \rightarrow C_*^\infty(S^1, \Omega K).$$

That  $\pi_2(\Omega K)$  is in fact central follows from the fact that it is a discrete normal subgroup of the connected group  $\mathcal{K}$ . For general, not necessarily simply connected  $G$ , we only obtain a crossed module

$$C_*^\infty(B^2, G)/C_*^\infty(S^2, G)_e \rightarrow C_*^\infty(S^1, G).$$

Since the image of this morphism is precisely  $C_*^\infty(S^1, G)_e$ , this in turn gives rise to the four term exact sequence

$$\pi_2(G) \rightarrow C_*^\infty(B^2, G)/C_*^\infty(S^2, G)_e \rightarrow C_*^\infty(S^1, G) \rightarrow \pi_1(G).$$

This sequence has a characteristic class in  $H^3(\pi_1(G), \pi_2(G))$ , which has first been constructed in [EM46].

The second smooth map, naturally associated to  $\mathcal{K}$  is given by  $\mathcal{K} \rightarrow \mathbb{R}$ ,  $[f] \mapsto \int_f \omega^l$ , where the integral only depends on the homotopy class of  $f$  because  $\omega^l$  is closed.

For the following lemma we define a generalisation of the cocycle  $\kappa$  by

$$\begin{aligned} \kappa_k : (P_e \Omega K) \times (P_e \Omega K) &\rightarrow U(1), \\ (\gamma, \eta) &\mapsto \exp \left( k \cdot \int_0^1 \int_0^1 \langle \gamma(s)^{-1} \gamma'(s), \eta'(t) \eta(t)^{-1} \rangle ds dt \right), \end{aligned}$$

which is for  $k = 1$  the cocycle  $\kappa$  from [Mur88] (cf. also [BCSS07]).

**PROPOSITION 4.3.** *For each  $k \in \mathbb{R}$ , the Lie group  $\mathcal{K}$  acts smoothly from the right on  $U(1) \times P_e \Omega K$  by*

$$(4.1) \quad (z, \gamma) \cdot [f] := (z \cdot \exp(-k \cdot \int_f \omega^l) \cdot \kappa_k(\gamma, f|_{\partial B^2}), \gamma \cdot f|_{\partial B^2}).$$

**PROOF.** It is clear that the action map is smooth on the product

$$(U(1) \times P_e \Omega K) \times \mathcal{K},$$

because the restriction map  $\mathcal{K} \rightarrow C_*^\infty(S^1, \Omega K)$  and the integration map  $\mathcal{K} \rightarrow U(1)$  are smooth.

In order to show that (4.1) actually defines a group action we have to verify that  $(z, \gamma) \cdot [f \cdot g] = ((z, \gamma) \cdot [f]) \cdot [g]$ , which is equivalent to

$$\begin{aligned} \exp(-k \cdot \int_{f \cdot g} \omega^l) \cdot \kappa_k(\gamma, (f \cdot g)|_{\partial B^2}) &= \\ \exp(-k \cdot \int_f \omega^l) \cdot \kappa_k(\gamma, f|_{\partial B^2}) \cdot \exp(-k \cdot \int_g \omega^l) \cdot \kappa_k(\gamma \cdot f|_{\partial B^2}, g|_{\partial B^2}). \end{aligned}$$

But this in turn follows immediately from the cocycle condition for  $\kappa_1$ , because

$$\kappa_1(f|_{\partial B^2}, g|_{\partial B^2}) = \exp(\int_f \omega^l) \cdot \exp(\int_g \omega^l) \cdot \exp(-\int_{f \cdot g} \omega^l)$$

follows from  $f|_{\partial B^2}, g|_{\partial B^2} \in C_*^\infty(S^1, \Omega K)$  (cf. [Mur88, Sect. 6]).  $\square$



For each  $k \in \mathbb{R}$ , the action (4.1) now defines an action Lie groupoid

$$(4.2) \quad (U(1) \times P_e \Omega K \times \mathcal{K} \rightrightarrows_k U(1) \times P_e \Omega K),$$

i.e.,  $s(z, \gamma, [f]) = (z, \gamma)$ ,  $t(z, \gamma, [f]) = (z, \gamma) \cdot [f]$  and

$$((z, \gamma) \cdot [f], [f']) \circ ((z, \gamma), [f]) = (z, \gamma, [f \cdot f']).$$

From formula (4.1) we see in particular, that the action of  $\mathcal{K}$ , and thus the Lie groupoid (4.2) is not proper for arbitrary  $k$ . In fact, the subgroup  $\mathbb{Z} \cong \pi_2(\Omega K) \subseteq \mathcal{K}$  acts on  $U(1) \times P_e \Omega K$  by

$$a.(z, \gamma) = (z \cdot \exp(-k \cdot a), \gamma),$$

since we assumed that  $\omega$  is normalised so that  $\int_\sigma \omega^l = 1$  for a generator  $[\sigma]$  of  $\pi_2(\Omega K) \cong \pi_3(K)$ . Of course, the interesting range for  $k$  in the previous proposition is  $k \in [0, 1]$ , for then the quotient of the action “interpolates” between the trivial and the universal extension:

$$\text{for } k = 0 : U(1) \times P_e \Omega K / \mathcal{K} \cong U(1) \times \Omega K$$

$$\text{for } k = 1 : U(1) \times P_e \Omega K / \mathcal{K} \cong \widehat{\Omega K}_1$$

Moreover, we see that for each  $k \in \mathbb{Q}$  the quotient  $U(1) \times P_e \Omega K / \mathcal{K}$  exists as a manifold, we shall give a precise argument for this at the end of the next section. However, the group structure on  $U(1) \times P_e \Omega K$ , given by the cocycle  $\kappa$ , only induces a group structure on the quotient in the case  $k \in \mathbb{Z}$ .

## 5. Lie groupoids as principal 2-bundles

In this section we show that the Lie groupoids, derived in the previous section, possess the structure of a principal 2-bundle. For this we give at first a very short and condensed introduction to principal 2-bundles. The details can be found in [Woc09b].

A strict Lie 2-group is a category object in the category of locally convex Lie groups i.e., it consists of two locally convex Lie groups  $G_0$  and  $G_1$ , together with morphisms  $s, t : G_1 \rightrightarrows G_0$ , a morphism  $i : G_0 \rightarrow G_1$  and a morphism  $c : G_1 \times_{s \times t} G_1 \rightarrow G_1$  (assuming that the pull-back  $G_1 \times_{s \times t} G_1$  exists), such that  $(G_0, G_1, s, t, i, c)$  constitutes a small category. In short, we write  $(G_1 \rightrightarrows G_0)$  for this (cf. [BL04] and [Por08]). A smooth 2-space is simply a Lie groupoid and similar to the case of Lie groups and manifolds, one defines a (right)  $(G_1 \rightrightarrows G_0)$ -2-space to be a 2-space  $(M_1 \rightrightarrows M_0)$ , together with a smooth functor

$$(\rho_1, \rho_0) : (M_1 \rightrightarrows M_0) \times (G_1 \rightrightarrows G_0) \rightarrow (M_1 \rightrightarrows M_0),$$

such that  $\rho_1$  defines a (right)  $G_1$ -action on  $M_1$  and  $\rho_0$  defines a (right)  $G_0$ -action on  $M_0$ . Similarly, one defines a morphism of  $(G_1 \rightrightarrows G_0)$ -2-spaces  $(M_1 \rightrightarrows M_0)$  and  $(N_1 \rightrightarrows N_0)$  to be a smooth functor  $(\varphi_1 \times \varphi_0) : (M_1 \times M_0) \rightarrow (N_1 \times N_0)$  such that  $\varphi_1$  (respectively  $\varphi_0$ ) defines a morphism of  $G_1$  (respectively  $G_0$ )-spaces. A 2-morphism  $\alpha : \varphi \Rightarrow \psi$  between two morphisms  $\varphi, \psi : (M_1 \times M_0) \rightarrow (N_1 \times N_0)$  of  $(G_1 \rightrightarrows G_0)$ -spaces consists of a smooth map  $\alpha : M_0 \rightarrow N_1$  such that  $\alpha$  defines a natural transformation between the functors  $\varphi$  and  $\psi$  and, moreover, satisfies  $\alpha(m.g) = \alpha(m). \text{id}_g$  for each  $m \in M_0$  and  $g \in G_0$ .

With this said one defines a principal  $(G_1 \rightrightarrows G_0)$ -2-bundle over the smooth manifold  $M$  (viewed as a smooth 2-space with only identity morphisms, we write

$\underline{M}$  for this 2-space) as follows. It is a smooth  $(G_1 \rightrightarrows G_0)$ -2-space  $(P_1 \rightrightarrows P_0)$ , together with a smooth functor  $\pi : (P_1 \rightrightarrows P_0) \rightarrow (M \rightrightarrows M)$ , commuting with the action functor  $\rho$ , such that there exist

- an open cover  $(U_i)_{i \in I}$  of  $M$
- morphisms

$$\begin{aligned} \Phi_i : \pi^{-1}(\underline{U}_i) &\rightarrow \underline{U}_i \times (G_1 \rightrightarrows G_0) \quad \text{and} \\ \overline{\Phi}_i : \underline{U}_i \times (G_1 \rightrightarrows G_0) &\rightarrow \pi^{-1}(\underline{U}_i) \end{aligned}$$

of  $(G_1 \rightrightarrows G_0)$ -2-spaces

- 2-morphisms

$$\begin{aligned} \tau_i : \Phi_i \circ \overline{\Phi}_i &\Rightarrow \text{id}_{\underline{U}_i \times (G_1 \rightrightarrows G_0)} \\ \overline{\tau}_i : \overline{\Phi}_i \circ \Phi_i &\Rightarrow \text{id}_{\pi^{-1}(\underline{U}_i)} \end{aligned}$$

between morphisms of  $(G_1 \rightrightarrows G_0)$ -2-spaces,

such that  $\pi$ ,  $\Phi_i$  and  $\overline{\Phi}_i$  commute in the usual way with the projection functor  $\text{pr} : \underline{U}_i \times (G_1 \rightrightarrows G_0) \rightarrow \underline{U}_i$ .

We are now aiming at showing that the action Lie groupoid

$$(U(1) \times P_e \Omega K \times \mathcal{K} \rightrightarrows_k U(1) \times P_e \Omega K)$$

possesses the structure of a principal 2-bundle (we used the subscript  $k$  to denote the value of  $k$  in the action map (4.1)). The structure 2-group of this bundle shall be given by  $(U(1) \times \pi_2(\Omega K) \rightrightarrows_k U(1))$  with  $s(z, [\sigma]) = z$ ,  $t(z, [\sigma]) = z \cdot \exp(-k \cdot \int_\sigma \omega^l)$  and  $(z \cdot \exp(-k \cdot \int_\sigma \omega^l), [\sigma']) \circ (z, [\sigma]) = (z, [\sigma' \cdot \sigma])$ .

Before showing the claim of this section, we have to pass from the action Lie groupoid (4.2) to a Morita equivalent one, which we will denote by  $(P_1 \rightrightarrows_k P_0)$ . For this we choose a system  $(\sigma_i : U_i \rightarrow P_e \Omega K)_{i \in I}$  of smooth local sections of the principal bundle  $\text{ev} : P_e \Omega K \rightarrow \Omega K$ . For technical reasons, that will become apparent later, we choose this system so that there exists smooth maps  $\sigma_{ij} : U_i \cap U_j \rightarrow \mathcal{K}$  such that  $\sigma_i(x) = \sigma_j(x) \cdot \sigma_{ij}(x)|_{\partial B^2}$ . Then we set

$$P_0 := \coprod_{i \in I} (U(1) \times \{\sigma_i(x) : x \in U_i\}),$$

which we endow with the smooth structure induced from  $U(1) \times P_e \Omega K$ . The set of morphisms we set to be

$$P_1 := \{(z, \gamma, \eta, [f]) \in U(1) \times P_0 \times P_0 \times \mathcal{K} : \text{ev}(\gamma) = \text{ev}(\eta), \gamma = \eta \cdot f|_{\partial B^2}\}.$$

For a fixed choice of  $\gamma$  and  $\eta$ , the possible different choices of  $[f]$  are parametrised by  $\pi_2(\Omega K)$ , and so  $P_1$  has a natural manifold structure, modelled on  $C_*^\infty(S^1, \Omega K)$ . Source and target maps are induced by the two projections from  $P_1$  to  $P_0$  and composition is induced by multiplication in  $\mathcal{K}$ .

We define a smooth functor from  $(P_1 \rightrightarrows_k P_0)$  to  $(U(1) \times P_e \Omega K \times \mathcal{K} \rightrightarrows_k U(1) \times P_e \Omega K)$  by inclusion on objects and on morphisms by  $(z, \gamma, \eta, [f]) \mapsto (z, \gamma, [f])$ . One easily checks that this functor actually defines a Morita equivalence.

There exists an obvious  $(U(1) \times \pi_2(\Omega K) \rightrightarrows_k U(1))$ -2-space structure on  $(P_1 \rightrightarrows_k P_0)$ , given by

$$(5.1) \quad (z, \gamma).w = (z \cdot w, \gamma) \quad \text{on objects and by}$$

$$(5.2) \quad (z, \gamma, \eta, [f]).(w, [\sigma]) = (z \cdot w \cdot \exp(-k \cdot \int_{\sigma}), \gamma, \eta, [f \cdot \sigma]) \quad \text{on morphisms.}$$

Moreover, there exists a natural smooth functor  $\pi : (P_1 \rightrightarrows_k P_0) \rightarrow \underline{\Omega K}$ , given on objects by  $(z, \gamma) \mapsto \text{ev}(\gamma)$  and on morphisms by  $(z, \gamma, \eta, [f]) \mapsto \text{ev}(\gamma)$ . We are now ready to prove the main result on this section.

**PROPOSITION 5.1.** *The  $(U(1) \times \pi_2(\Omega K) \rightrightarrows_k U(1))$ -2-space structure on  $(P_1 \rightrightarrows_k P_0)$ , given by (5.1) and (5.2), along with the smooth functor  $\pi$ , defines a principal 2-bundle.*

**PROOF.** We observe that  $(z, \gamma)$  is an object of  $\pi^{-1}(\underline{U}_i)$  if and only if  $\text{ev}(\gamma) \in U_i$  and  $\gamma = \sigma_i(\text{ev}(\gamma))$ . From  $\text{ev}(\gamma) = \text{ev}(\gamma \cdot f|_{\partial B^2})$  for each  $[f] \in \mathcal{K}$  it follows that a morphism has source in  $\pi^{-1}(\underline{U}_i)$  if and only if it has target in  $\pi^{-1}(\underline{U}_i)$ , so that  $\pi^{-1}(\underline{U}_i)$  is in fact a full subcategory.

We now define local trivialisations  $\Phi_i$  by

$$\begin{aligned} (z, \gamma) &\mapsto (\text{ev}(\gamma), z) && \text{on objects and by} \\ (z, \gamma, \eta, [f]) &\mapsto (z, \sigma_{ij}(\gamma, \eta) \cdot [f]^{-1}) && \text{on morphisms.} \end{aligned}$$

This is smooth due to the requirements that we put on the choice of  $(\sigma_i : U_i \rightarrow P_e \Omega K)_{i \in I}$  and that it actually defines a functor follows from the fact that  $\pi_2(\Omega K)$  is central in  $\mathcal{K}$ . The “inverse” trivialisations  $\overline{\Phi}_i$  we define by

$$\begin{aligned} (l, z) &\mapsto (z, \sigma_i(l)) && \text{on objects and by} \\ (l, (z, [\sigma])) &\mapsto (z, \sigma_i(l), \sigma_i(l), e) && \text{on morphisms.} \end{aligned}$$

These obviously define smooth functors commuting with the  $(U(1) \times \pi_2(\Omega K) \rightrightarrows_k U(1))$ -action, and we have  $\Phi_i \circ \overline{\Phi}_i = \text{id}$ . We then define  $\overline{\tau}_i : \overline{\Phi}_i \circ \Phi_i \Rightarrow \text{id}$  by

$$(z, \gamma) \mapsto (z, \sigma_i(x), \sigma_j(x), \sigma_{ij}(x)) \text{ if } \gamma = \sigma_j(x) \text{ for } x \in U_{ij}.$$

It is easily checked that  $\overline{\tau}_i$  actually defines a natural transformation and satisfies  $\overline{\tau}_i((z, \gamma).z') = \overline{\tau}_i((z, \gamma)).(z', e)$ .  $\square$

Note that the functors  $\Phi_i$  and the natural transformations  $\overline{\tau}_i$  in the previous proof were smooth for they only need to be defined if  $(\gamma, \eta)$  can be written as  $(\sigma_i(x), \sigma_j(x))$  for  $x = \text{ev}(\gamma) = \text{ev}(\eta)$ . If one tried to define a 2-bundle structure on the whole action groupoid (4.2) in a similar way, then one would need a smooth global section of  $\mathcal{K} \rightarrow C_*^\infty(S^1, \Omega K)$ , which does not exist. Thus the passage to the Morita equivalent groupoid  $(P_1 \rightrightarrows_k P_0)$  was necessary to ensure the smoothness properties of the local trivialisations.

**COROLLARY 5.2.** *If  $k \in \mathbb{Q}$ , then the quotient  $P_k$  of the groupoid  $(P_1 \rightrightarrows_k P_0)$  can be endowed with the structure of a smooth manifold. Moreover, the action (5.1) induces on  $P_k$  the structure of a smooth  $(U(1)/k)$ -principal bundle.*

**PROOF.** This is exactly the construction of the band of a principal 2-bundle from [Woc09b].  $\square$

The previous result can also be obtained as in Section 2 by considering the Lie group  $U(1)/k = \mathbb{R}/(\mathbb{Z} + k\mathbb{Z})$ . This shows actually that  $P_k$  can also be endowed with a Lie group structure, turning

$$(U(1)/k) \rightarrow P_k \rightarrow \Omega K$$

into a central extension of Lie groups. However, the group structure on  $P_k$  is not induced by the one on  $U(1) \times_{\kappa_1} P_e \Omega K$  any more.

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